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MINIMAL DIFFERENTIAL IDENTITIES IN PRIME RINGS

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ABSTRACT

It is known that a prime ring which satisfies a polynomial identity with derivations applied to the variables must satisfy a generalized polynomial identity, but not necessarily a polynomial identity. In this paper we detemine the minimal identity with derivations which can be satisfied by a non-PI prime ring R. The main result shows, essentially, that this identity is the standard identity S_3 with D applied to each variable, where $D = \mathrm{ad}(y)$ for y in R, $y^2 = 0$, and y of rank one in the central closure of R.

A prime ring which satisfies a polynomial identity certainly satisfies the same identity with derivations acting on the variables. On the other hand, if a prime ring satisfies a polynomial identity with derivations applied, then it follows from work of V. K. Kharchenko [5] that the ring must satisfy a generalized polynomial identity, but need not satisfy any polynomial identity, by an example of A. Kovacs [6]. The purpose of this paper is to determine the minimal polynomial identities with derivation which can be satisfied by a prime ring not satisfying a polynomial identity. Now since any such prime ring does satisfy a generalized polynomial identity, it follows from a well known result of W. S. Martindale [8] that the central closure of the ring is primitive with nonzero socle and its associated division ring is finite dimensional over its center. Thus, it is not surprising that our study of minimal polynomial identities with derivations reduces to the consideration of such identities for primitive rings with nonzero socle. By careful calculation in such rings we can show that any minimal polynomial identity with derivations is a sum of identities $S_3(x^d, y^d, z^d)$ for d an inner derivation determined by an element of rank one and square zero, where $S_3(x, y, z)$ is the standard identity of degree three.

We begin with some notation and a discussion of what we mean by "minimal

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polynomial identity with derivations." Throughout the paper R will denote a prime ring with extended centroid C (see [8]). Let $C\{X\}$ be the free algebra over C in a set of indeterminates X, indexed by the positive integers, and let $\mathrm{Der}(R)$ be the Lie ring of derivations of R. For any $p \in C\{X\}$ and suitable n, p lies in the subalgebra $C\{x_1, \ldots, x_n\}$ and we write $p = p(x_1, \ldots, x_n)$. Call $p(x_1, \ldots, x_n) \in C\{X\} - \{0\}$ a *polynomial differential identity*, or PDI, for R if for $k \leqq n$ there are $d_1, \ldots, d_k \in \mathrm{Der}(R) - \{0\}$, so that for all $(r_1, \ldots, r_n) \in R^n$,

$$p(r_1^{d_1}, \ldots, r_k^{d_k}, r_{k+1}, \ldots, r_n) = 0,$$

where r^d denotes the image of r under d. When it is convenient to do so, we will consider *a* PDI p to be a function of the variables $\{x_1^{d_1}, \ldots, x_k^{d_k}, x_{k+1}, \ldots, x_n\}$.

Given a PDI $p(x_1^{d_1}, \ldots, x_k^{d_k}, x_{k+1}, \ldots, x_n)$ for R, let $\deg(p)$ be the usual degree of p, and let D-$\deg(p)$ denote the degree of p in those variables $\{x_1, \ldots, x_k\}$ to which derivations have been applied. We shall say that p is of type (D-$\deg(p), \deg(p)$), and we order types by setting $(k, n) < (k', n')$ if either $k < k'$ or if $k = k'$ and $n < n'$. If R satisfies a polynomial identity, then R satisfies a PDI of type $(0, n)$. A PDI p for R is minimal if it is of minimal type among all PDIs for R. Our goal is to find what type is minimal over all non-PI prime rings R, and then to describe the PDIs of this type. It will turn out that the description is the same if we had defined minimal type by first minimizing degree, and then D-degree.

Any PDI for R can be linearized in the usual way to get a PDI which is linear in each variable. Although the linearization may increase the number of variables which have a derivation applied, it does not increase the type of the original PDI. Furthermore, if $p(x_1, \ldots, x_n)$ is such a linearized PDI for R, then so are both $p(x_1, \ldots, x_{n-1}, 0)$ and $p - p(x_1, \ldots, x_{n-1}, 0)$. Repetition of this procedure for arbitrary subsets of $\{x_1, \ldots, x_n\}$ shows that p may be written as a sum of PDIs for R, each of which is multilinear, homogeneous, and of type not exceeding the type of p. Consequently, in determining what the minimal type is, we may restrict our attention to PDIs which are multilinear and homogeneous. Henceforth, our use of the term multilinear PDI will mean both multilinear and homogeneous.

Next we give some explicit examples of multilinear PDIs satisfied by non-PI prime rings. These examples are crucial to our investigation since they severely limit the types of PDIs which can be minimal. Our first example is one discussed by Kovacs [6] who used it to produce a multilinear PDI of type $(4p+1, 4p+1)$ in characteristic p, and of type $(9,9)$ in characteristic zero. By using a different approach one easily obtains a characteristic free example of smaller type.

EXAMPLE 1. Let R be the ring of countable by countable row finite matrices over a field F, and let d be the inner derivation of R given by $r^d = [r, e_{11}]$, where e_{11} is the usual upper left matrix unit. Regarding R as the ring of linear transformations acting on the right of a countable dimensional vector space over F, it is clear that r^d has rank at most two for every $r \in R$. A straightforward computation shows that r^d satisfies a cubic polynomial $x^3 + ax \in F[x]$. It follows that R satisfies the PDI $[[(x_1^d)^3, x_2], [x_1^d, x_2]]$ of degree six. Linearizing this expression gives a multilinear PDI of type $(4, 6)$, independent of the characteristic of F.

We note that one could mimic the construction in Example 1 when F is replaced by a division ring D finite dimensional over its center F. In that case, r^d is algebraic over F of bounded degree, so by commutation one could again obtain a PDI for R of rather large degree [4; p. 230]. Also, although our multilinear PDI is of smaller type than the one obtained by Kovacs [6], we should point out that he was interested in obtaining a standard polynomial. Our next example shows that one can find a PDI by using $S_3(x, y, z)$, and represents what will turn out to to be the typical (multlinear) PDI of minimal type.

EXAMPLE 2. Let R be as in Example 1 and choose $a \in R$ so that $a^2 = 0$ and aR is a minimal right ideal of R. For example, one could take $a = e_{12}$. Now for some $e^2 = e$, $aR = eR$, so $ea = a$ and $eRe \cong eF$ is a field. It follows that for $r, s \in R$, $arasa = asara$. Let d be the inner derivation defined by $r^d = [r, a]$, and let S_3 be the standard polynomial of degree three. Then $S_3(x_1^d, x_2^d, x_3^d)$ is a multilinear PDI for R. The verification of this fact is fairly easy. Observe that $S_3(ra - ar, sa - as, ta - at)$ is the product $(ra - ar)(sa - as)(ta - at) = arasta - arsata - arasat + rasata$, together with similar expressions in each permutation of r, s, and t with the appropriate sign change. Using $axaya = ayaxa$ it is straightforward to show that all terms cancel.

In view of Example 2, we know that if p is a PDI of minimal type (k, n) over all non-PI prime rings, then $k \leqq 3$. In fact, we shall show that no PDI for a non-PI prime ring can have type (k, n) with $k < 3$. We want to reduce to the case where R is a simple ring equal to its socle. Using the structure theory of such rings we can make the computations necessary to obtain the results we seek.

Recall that C is the extended centroid of R, and let Q be the Martindale quotient ring of R (see [8]). One may regard Q as equivalence classes of left R-module homomorphisms from ideals of R to R. For $q \in Q$ there is a nonzero ideal J of R with $Jq \subset R$, and if $Jq = 0$ then $q = 0$. It follows that R embeds in Q as right multiplications, and that C is the center of Q. It is easy to show that each

$d \in \mathrm{Der}(R)$ extends to a derivation of Q ([5; p. 156] or [7]), so also extends to a derivation of C. Thus if p is a PDI for R, then each $d_i \in \mathrm{Der}(R)$ appearing in p can be considered as a derivation of the prime ring $RC \subset Q$. Our first lemma implies that $(C)d_i = 0$ if p is of minimal type.

LEMMA 1. *If $p(x_1^{d_1}, \ldots, x_k^{d_k}, x_{k+1}, \ldots, x_n)$ is a multilinear PDI for R, then either $(C)d_i = 0$ for each $i \leqq k$, or R satisfies a PDI of smaller type.*

PROOF. Suppose that $y^{d_i} \neq 0$ for some $i \leqq k$ and some $y \in C$. The definition of Q gives a nonzero ideal J of R satisfying $Jy \subset R$. Choose $r_i \in Jy$ and $r_j \in R$ for $j \neq i$, and evaluate p to obtain $y^{d_i} q(r_1^{d_1}, \ldots, r_n) = 0$, where q is p with $x_i^{d_i}$ replaced by x_i. Since $y^{d_i} \neq 0$, it suffices to show that q is a PDI for R, since q is of type $(k-1, n)$. We know that substitutions into q give zero, but only if x_i is replaced by an element of J. Substituting arbitrarily for the other variables in q gives a linear polynomial $h(x) = \Sigma\, a_t x b_t$, where all $a_t, b_t \in RC \cup \{1\}$. Now $h(J) = 0$ so it follows ([3; proof of Lemma 1.3.2, pp. 22–23] or [7; Lemma 1]) that $h(x)$ is actually an identity for R. Therefore, q is a PDI for R, proving the lemma.

An immediate consequence of Lemma 1 is that any multilinear PDI of minimal type satisfied by R is also satisfied by RC. Now if R satisfies a multilinear PDI, then R satisfies a nonzero generalized polynomial identity [7; Theorem 8], [5; Corollary 5, p. 163], so RC is a primitive ring with nonzero socle and underlying division ring which is finite dimensional over C [8; Theorem 3, p. 579]. Therefore, a multilinear PDI of minimal type can be considered as a PDI for RC, as just described, and $C^{d_i} = 0$ for each d_i appearing. Let H be the socle of RC and note that H is a simple algebra with centroid C, and H is infinite dimensional over C, unless R satisfies a polynomial identity. Also, if eH is a minimal right ideal of H, then eHe is a finite dimensional division algebra over eC. Since $H^2 = H$, it follows easily that $H^d \subset H$ for any $d \in \mathrm{Der}(RC)$, so any multilinear PDI of minimal type for R is satisfied by H. Finally, if F is an algebraic closure of C, then $H \otimes_C F$ is a simple ring equal to its socle and has (extended) centroid F. Furthermore, if $d \in \mathrm{Der}(H)$ and $C^d = 0$ then d extends to a derivation of $H \otimes F$ by extending $(h \otimes a)^d = h^d \otimes a$, and so $H \otimes F$ satisfies any multilinear PDI satisfied by H. Consequently, to determine what type is minimal, it suffices to assume that R is a special ring as defined next.

DEFINITION. A ring R is *special* if it is a non-Artinian simple ring which is equal to its socle, and if for every minimal idempotent $e \in R$, $eRe \cong F$, the centroid of R.

The computations we will make in studying multilinear PDIs of minimal type require a number of standard or easy facts about non-Artinian simple rings which equal their socles, and we review these now. Let R be such a ring, eR a minimal right ideal, $D \cong eRe$ the associated division ring, and F the centroid of R. Of course, when R is special, $D = F$. The statements we make are about right ideals, but each has an obvious analogue for left ideals. Also, for any nonempty subset B of R, $l(B)$ and $r(B)$ denote the left and right annihilators, respectively, of B in R, and $\mathrm{Ann}(B) = l(B) \cap r(B)$. Now R is completely reducible as a right R-module, so every right ideal is a direct sum of minimal right ideals. If a right ideal T is the direct sum of finitely many minimal right ideals, then the number of summands is unique and will be denoted by $\dim(T)$. In particular, $\dim(rR)$ is finite for any $r \in R$. If T is a right ideal of R with $\dim(T)$ finite, then $T = eR$, for e an idempotent, $l(T) = R(1-e) = \{r - re \mid r \in R\}$, and $r(l(T)) = T$. For any nonzero $e = e^2 \in R$, $eRe \cong M_n(D)$ for $n = \dim(eR)$, and $\mathrm{Ann}(e)$, sometimes written $(1-e)R(1-e)$, is isomorphic to R. A result which is critical for our calculations is Litoff's theorem [4; Theorem 3, p. 90] which we state now in the form most useful to us.

THEOREM (Litoff). *If R is a simple ring equal to its socle, then for $\{a_1, \dots, a_n\} \subset R$ there is an idempotent $e \in R$ so that $\{a_1, \dots, a_n\} \subset eRe$.*

Our next Lemma is an easy consequence of Litoff's theorem and will be convenient to have for reference.

LEMMA 2. *Let R be a non-Artinian simple ring equal to its socle. If B is a finite subset of R, then* $\mathrm{Ann}(B)$ *contains an infinite set of orthogonal idempotents.*

PROOF. By Litoff's theorem, $B \subset eRe$, and since R is isomorphic to $\mathrm{Ann}(e) \subset \mathrm{Ann}(B)$, it suffices to prove that R contains an infinite set of orthogonal idempotents. This fact is easy to obtain by using Zorn's Lemma and Litoff's theorem again.

A number of useful facts and observations concerning right ideals and derivations are combined in our next lemma. For $B \subset R$ and $d \in \mathrm{Der}(R)$ we write $B^d = \{b^d \mid b \in B\}$.

LEMMA 3. *Let R be a non-Artinian simple ring equal to its socle. If T is a nonzero right ideal of R and if d is a nonzero derivation of R, then the following hold:*

(1) $r(R^d) = l(R^d) = 0$;

(2) $T^d = 0$ *implies* $T = 0$;

(3) *$T^dx = 0$ for $x \in R$, implies either $x = 0$ or $T^dT = 0$;*

(4) *if* $\dim(T)$ *is finite and $T^dT = 0$, then $T^d \subset T$;*

(5) *$T^d \subset T$ implies $l(T)^d \subset l(T)$;*

(6) *$T + T^d$ is a right ideal of R; and*

(7) *if* $\dim(T)$ *is finite then* $\dim(T + T^d)$ *is finite.*

PROOF. (1) If $R^da = 0$, then for $x, y \in R$, $0 = (xy)^da = x^dya$, so the primeness of R forces $a = 0$, since $d \neq 0$. Thus $r(R^d) = 0$, and $l(R^d) = 0$ follows from a similar argument.

(2) If $r \in R$ and $t \in T$ then $0 = (tr)^d = tr^d$, resulting in $T \subset l(R^d) = 0$ using (1).

(3) If $s, t \in T$, then $0 = (st)^dx = s^dtx$. Since R is a prime ring, either $T^dT = 0$ or $x = 0$.

(4) Choose $x, y \in T$ and $r \in R$, write $0 = (xr)^dy = x^dry + xr^dy$ and multiply on the left by any element in $l(T)$ to obtain $l(T)T^dRT = 0$. Using the primeness of R, one must conclude that $T^d \subset r(l(T)) = T$, since $\dim(T)$ is finite.

(5) Apply d to $l(T)T = 0$.

(6) This follows from $t^dr = (tr)^d - tr^d$.

(7) Since $T = eR$, $T^d = (eR)^d \subset eR + e^dR$.

Our last preliminary result gives a condition concerning the action of a derivation on right ideals which forces the derivation to be zero.

LEMMA 4. *Let R be a non-Artinian simple ring equal to its socle. If $d \in$* $\mathrm{Der}(R)$, *then the following are equivalent:*

(1) *for each minimal right ideal T of R, $T^d \subset T$;*

(2) *for each right ideal T of R, $T^d \subset T$; and*

(3) *$d = 0$.*

PROOF. Since every right ideal of R is a sum of minimal right ideals, it is clear that (1) implies (2), so it suffices to prove that (2) implies (3). Let e be any nonzero idempotent in R and set $T = r(Re)$. By assumption $T^d \subset T$ so Lemma 3 yields $(Re)^d = (l(T))^d \subset l(T) = Re$. Of course, $(eR)^d \subset eR$ also holds by assumption. Consequently $e^d \in eR \cap Re = eRe$, so that $e^d = ee^de = e(e^2)^de = ee^de + ee^de = 2e^d$, resulting in $e^d = 0$. But R is a simple non-Artinian ring so is generated by its idempotents [2; Chapter 1], so $R^d = 0$, forcing $d = 0$.

We are now in a position to examine a multilinear PDI of minimal type which is satisfied by a non-Artinian simple ring with minimal right ideal. The next few results will show that the type must be (3, 3); in fact, that no multilinear PDI can be of type $(1, n)$ or $(2, n)$. The first of these is a lemma which gives some important information abut which variables must have derivations applied.

LEMMA 5. *Let R be a non-Artinian simple ring equal to its socle, and* $p(x_1^{d_1}, \ldots, x_k^{d_k}, x_{k+1}, \ldots, x_n)$ *a multilinear PDI (of type (k, n)) satisfied by R. Then either each monomial of p begins and ends with an element in* $\{x_1^{d_1}, \ldots, x_k^{d_k}\}$, *or R satisfies a multilinear PDI of smaller type.*

PROOF. We prove only that each monomial of p begins with some $x_i^{d_i}$, since a similar argument would show that each monomial ends with such a variable. If the lemma were false, then $n > 1$ and one can write

$$p = \sum_{i=1}^{k} x_i^{d_i} q_i + \sum_{i=k+1}^{n} x_i p_i \qquad \text{where some } p_j \neq 0.$$

For $i \neq j$ choose $r_i \in R$ and use Lemma 2 to find a nonzero idempotent $e \in l(B)$, where $B = \{r_i^{d_i} \mid i \leqq k\} \cup \{r_i \mid i > k \text{ and } i \neq j\}$. Evaluating p with r_i replacing x_i and er replacing x_j, for any $r \in R$, yields $0 = ep = erp_j(r_1^{d_1}, \ldots, r_n)$. But R is a prime ring, $e \neq 0$ and $\{r_i\} \subset R$ is arbitrary, so p_j is a multilinear PDI for R of type $(k, n-1)$, contradicting the minimality of (k, n) and thereby proving the lemma.

An immediate consequence of Lemma 5 is that there is no PDI for R of type $(1, n)$, which we record as our next lemma.

LEMMA 6. *If R is a non-Artinian simple ring equal to its socle which satisfies a PDI of type (k, n), then $k > 1$.*

Lemma 5 puts strong restrictions on the form of a multilinear PDI of minimal type. The next step in our investigation is to show that the derivations appearing on the "ends" of the monomials of the PDI are inner derivations. To do this, we need a general result which gives a criterion on the kernel of a derivation which forces the derivation to be inner.

THEOREM 1. *Let R be a non-Artinian simple ring with centroid F, with $R = \mathrm{Soc}(R)$, and with associated division ring which is finite dimensional over F. If there are $d \in \mathrm{Der}(R)$ and an idempotent $e \in R$ so that $f^d = 0$ for every idempotent $f \in \mathrm{Ann}(e)$, then $r^d = [r, a]$ for $a \in eRe$.*

PROOF. Since $\mathrm{Ann}(e)$ is isomorphic to R, it is generated as a ring by its idempotents, from which $(\mathrm{Ann}(e))^d = 0$ follows. Now $e(\mathrm{Ann}(e)) = (\mathrm{Ann}(e))e = 0$, so applying d shows that $e^d \in \mathrm{Ann}(\mathrm{Ann}(e)) = eRe$, or equivalently, d induces a derivation on eRe. Thus $e^d = 0$, since e is the identity of eRe, and also the extension of d to F is zero, since $F\,\mathrm{Ann}(e)\subset \mathrm{Ann}(e)\subset \mathrm{Ker}(d)$. It follows that $(Fe)^d=0$. But Fe is the center of the simple Artinian ring eRe, so we may conclude that the restriction of d to eRe is the inner derivation induced by $A\in eRe$. Now $h=d-ad(A)\in \mathrm{Der}(R)$, where $(r)adA=rA-Ar$, and $(eRe)^h=(\mathrm{Ann}(e))^h=0$.

Next, we compute the effect of h on $el(e)=eR(1-e)$. Let $g^2=g\in \mathrm{Ann}(e)-\{0\}$ and consider $exg\in eR(1-e)$. Certainly $e+g$ is an idempotent, $exg\in(e+g)R(e+g)=W$, and $W^h\subset W$. As above, W is a simple Artinian ring and $(Z(W))^h=(F(e+g))^h=0$, so the restriction of h to W is $\mathrm{ad}(B)$, for some $B\in W$. Using $e^h=g^h=0$ and $e,g\in W$ yields $B=eBe+gBg$, and then using $(eRe)^h=(gRg)^h=0$ results in $B=c_1e+c_2g$ for $c_1,c_2\in F$. It follows that $(exg)^h=[exg,B]=(c_2-c_1)exg$, and so we may write $(erg)^h=f(g)erg$ for any $r\in R$. If $g'\in \mathrm{Ann}(\{e,g\})$ is a nonzero idempotent, then by what we have just shown,

$$f(g+g')er(g+g')=(er(g+g'))^h=(erg)^h+(erg')^h=f(g)erg+f(g')erg',$$

from which $f(g)=f(g+g')=f(g')$ results. Now for any nonzero idempotents $g_1,g_2\in \mathrm{Ann}(e)$, there is a nonzero idempotent $g_3\in \mathrm{Ann}(\{e,g_1,g_2\})$, so $f(g_1)=f(g_3)=f(g_2)$. Therefore, if $x\in eR(1-e)$ and $Rx=Rg$ for $g^2=g$, then $x=erg$ and $x^h=fx$, where $f\in F$ is independent of $x\in eR(1-e)$. A similar argument shows that $y^h=f'y$ for each $y\in(1-e)Re$.

Choose $x,y\in R$ so that $x\in(1-e)Re$, $y\in eR(1-e)$, and $xy\neq 0$. Then $xy\in \mathrm{Ann}(e)$, so $0=(xy)^h=x^hy+xy^h=(f+f')xy$, resulting in $f=-f'$. But now $x^h=[x,f'e]$ for $x\in(1-e)Re\cup eR(1-e)$, and so $h=\mathrm{ad}(f'e)$. Consequently, $d=\mathrm{ad}(A+f'e)$, for $A+f'e\in eRe$, completing the proof of the theorem.

Theorem 1 will be very useful, enabling us to conclude that all the derivations appearing in a PDI of minimal type are inner. Our next lemma will be used to eliminate PDIs of type $(2,n)$ and illustrates the computation which leads to the application of Theorem 1.

LEMMA 7. *Let R be a non-Artinian simple ring and p a multilinear PDI of minimal type among PDIs satisfied by R. If x^d either begins or ends a monomial appearing in p, then d is an inner derivation of R.*

PROOF. From Lemma 1 and the comments following Lemma 1, we may conclude that $C^d=0$, for C the centroid of R, that $R=\mathrm{Soc}(R)$, and that the division ring associated to R is finite dimensional over C. We will assume that x^d

starts a monomial; the proof in the other case is similar. Write $p = x^d p_1 + q$, note that $p_1 \neq 0$ by the choice of d and Lemma 5, and observe that p_1 is not a PDI for R because it is of smaller type than p. For convenience, let $p_1 = p_1(y_1^{d_1}, \ldots, y_k^{d_k}, y_{k+1}, \ldots, y_n)$ and choose $r_1, \ldots, r_n \in R$ so that substitution of r_i for y_i into p_1 does not result in zero. Set $B = \{r_1^{d_1}, \ldots, r_k^{d_k}, r_{k+1}, \ldots, r_n\}$ and use Litoff's Theorem first to find an idempotent e with $B \subset eRe$, and again to get $e, e^d \in fRf$. If $T = r(f)$, then for $t \in T$, $et = e^d t = 0$ and so $et^d = 0$. Substituting the r_i for the y_i and t for x in p results in $T^d p_1(r_1^{d_1}, \ldots, r_n) = 0$. The choice of $\{r_i\}$, together with Lemma 3, yields $T^d T = 0$. Use Litoff's Theorem again to write $f, f^d \in gRg$, and consider any idempotent $v \in \mathrm{Ann}(g)$. Now $v \in \mathrm{Ann}(f) \subset T$, so $v^d T = 0$, or equivalently, $v^d \in l(r(f)) = Rf$, which means that $v^d f = v^d$. But $v \in \mathrm{Ann}(f) \cap \mathrm{Ann}(f^d)$, and this implies that $v^d \in \mathrm{Ann}(f)$. Consequently, $v^d = 0$, so by applying Theorem 1 to d and the idempotent g, we may conclude that d is an inner derivation of R.

Our final lemma uses our previous results to show that the minimal types must be greater than $(2, n)$, for any n.

LEMMA 8. *Let R be a non-Artinian simple ring equal to its socle. If R satisfies a multilinear PDI of type (k, n), then $k \geqq 3$.*

PROOF. By Lemma 6, $k \geqq 2$, so it suffices to show that $k = 2$ is impossible. Suppose that R satisfies a multilinear PDI of type $(2, n)$, with $n > 2$ and n minimal. Using Lemma 5, write such a PDI as

$$p = x^d p_1(x_3, \ldots, x_n) y^h + y^h p_2(x_3, \ldots, x_n) x^d.$$

Note that neither p_1 nor p_2 is zero by Lemma 3. Let $T = eR$ be a minimal right ideal of R, set $L = l(T^d) \supset l(T + T^d) = l(fR)$, and observe that L contains an infinite set of orthogonal idempotents by Lemma 2, where $B = \{f\}$ in that lemma. Evaluate p using $x \in T$, $y \in R$, and $x_i \in L$ for $i \geqq 3$, obtaining $T^d p_1(L) R^h = 0$. Using Lemma 3 we conclude first that $T^d p_1(L) = 0$, and then that either $p_1(L) = 0$ or $T^d \subset T$. Suppose first that $p_1(L) = 0$; then the ring $L/((L \cap r(L))$ satisfies the polynomial identity p_1. Since R is a simple ring equal to its socle, $L/((L \cap r(L))$ is a simple ring which must be finite dimensional over its center by Kaplansky's Theorem [4; Theorem 1, p. 226]. However, from our observation above, it follows that $L/(L \cap r(L))$ must contain an infinite set of orthogonal idempotents. This contradiction shows that $p_1(L) \neq 0$. Hence, $T^d \subset T$ for each minimal right ideal T of R, so Lemma 4 forces $d = 0$.

Our argument has shown that if p is a multilinear PDI for R of type $(2, n)$, for n minimal, then $n = 2$. Consequently, $p = x^d y^h + c y^h x^d$, for some $c \in C$. Since p

has minimal type among all PDIs satisfied by R, Lemma 7 allows us to write

$$p = (xa - ax)(yb - by) + c(yb - by)(xa - ax) \qquad \text{for } a, b \in R.$$

Substitute into p, replacing y with any $r \in R$, and $x \in T = r(Ra + Rb + Rr)$ to get $Ta(rb - br) = 0$. The primeness of R gives $aR^h = 0$, so either $a = 0$, which means $d = 0$, or $h = 0$ by Lemma 3. This contradiction to the definition of PDI proves the lemma.

We now have all of the pieces needed to prove our main result, for special rings. From Lemma 8 we known that any multilinear PDI for a special ring must have type (k, n) with $k > 3$, and from Example 2, there is a multilinear PDI of type $(3, 3)$. By linearizing, this shows that any PDI of minimal degree must be of degree 3. We now show that there is essentially only one possibility for a PDI of minimal type (or degree).

Theorem 2. *Let R be a special ring and p a PDI for R of minimal type (k, n). Then $n = k = 3$, p is linear in each variable, and $p = p_1 + \cdots + p_n$ where each p_i is a multilinear PDI for R of type $(3, 3)$ and satisfies:*

(1) *$p_i = c_i S_3(x_i, y_i, z_i)$ is of type $(3, 3)$, where $c_i \in F$;*

(2) *the three derivations asociated with p_i are all F-multiples of one another; and*

(3) *each derivation associated with p_i has the form* ad(a) *for $a \in R$ with $a^2 = 0$ and aR a minimal right ideal of R.*

Proof. We begin by reducing to the multilinear case, using arguments similar to those in [1]. As in our earlier comments about linearization, if $p = p(x_1, \ldots, x_n)$ then clearly both $p(x_1, \ldots, x_{n-1}, 0)$ and $p - p(x_1, \ldots, x_{n-1}, 0)$ are PDIs for R. It follows easily that $p = p_1 + \cdots + p_t$ where each variable in p_i appears in each monomial of p_i, and each p_i is a PDI for R, so is of type (k, n). Thus it suffices to assume that every variable of p appears in each monomial of p, and to show that p satisfies (1), (2), and (3) of the theorem. Suppose that $k = D - \deg p \leqq 2$. The linearization of p gives a multilinear PDI for R of type $(2, n)$, contradicting Lemma 8. Thus, $k \geqq 3$, and so $k = n = 3$ follows from the fact that p is of minimal type and from Example 2 showing the existence of a PDI for R of type $(3, 3)$. In particular, because of our first reduction, p has at most three variables, and is multilinear if it has exactly three variables. Assume now that the theorem holds for multilinear PDIs of minimal type, but that either $p = p(x)$, or $p = p(x, y)$. Suppose that char $R \neq 2$. In the first case, $cp = x^3 + ax^2 + bx$ for $a, b, c \in F$, and the linearization of cp, which is the linearization of x^3, gives a multilinear PDI for R of type $(3,3)$ which is not an F-multiple of $S_3(x, y, z)$. This contradicts the theorem in the multilinear case, so $p = p(x, y)$. By symmetry, we may assume that p has a nonzero monomial of degree two in x and degree one in y. But now, linearizing p on x gives a multilinear PDI for R of type $(3,3)$ which is not an F-multiple of $S_3(x, y, z)$.

Now assume that char $R = 2$, so that linearizing $cp = x^3 + ax^2 + bx$ gives the multilinear PDI $S_3(x, y, z)$ for R. Since this is of minimal type, our assumption that the theorem holds in the multilinear case yields $d = \mathrm{ad}(r)$ for $r^2 = 0$, and also that $r = erf$ for e and f primitive orthogonal idempotents. Clearly, $f^d = r$, so $p(f) = 0$ means $br = 0$, forcing $b = 0$. Choose $s \in R$ with $fse \neq 0$. Then $(fse)^d = c_1 f + c_2 e$ for $c_i \in F - \{0\}$, so $p(fse) = c_1^3 f + c_2^3 e + a(c_1^2 f + c_2^2 e)$. It follows that $a = c_1 = c_2$. If $F \neq \mathrm{GF}(2)$ and $c_3 \in F - \{0, 1, c_1\}$, then $p(c_3 fse) = 0$ results in $a = c_3 c_1$. This contradiction shows that $F = \mathrm{GF}(2)$, so $a = 1$, and gRh has exactly one nonzero element for any primitive idempotents g and h. Since R is a special ring, we may choose a primitive idempotent g orthogonal to both e and f, and choose $s_i \in R$ so that $t = fs_1 g + gs_2 e \neq 0$. But now $0 = p(t) = (t^d)^3 + (t^d)^2 = erfs_1 gs_2 erf$, which cannot be zero by choice of the s_i. This contradiction shows that $p \neq p(x)$.

Turning to the two variable case, assume $cp = a_1 x^2 y + a_2 xyx + a_3 yx^2 + p_1$, where p_1 has degree two in y, and some $a_i \neq 0$. Linearizing on x shows that $a_1(xzy + zxy) + a_2(xyz + zyx) + a_3(yxz + yzx)$ is a multilinear PDI for R of minimal type. As above, assuming the theorem for multilinear PDIs implies that $a_1 = a_2 = a_3 \neq 0$ and that $d = \mathrm{ad}(r)$ with $r^2 = 0$ and $r = erf$. Linearizing cp, and so p_1 on y shows that the coefficients of p_1 are all equal, so we may assume $cp = q(x, y) + aq(y, x)$ for $q(x, y) = x^2 y + xyx + yx^2$ and $a \in F$. Since e and f are primitive, and R is special, there is $t = fse$ so that $t^d = tr + rt = f + e$. Note that $f^d = r$, so that $0 = cp(t, f) = q(t, f) + aq(f, t) = r$. Hence, we must conclude that $p \neq p(x, y)$. Therefore, the theorem holds in general if we can prove it when p is multilinear of type $(3,3)$.

By re-ordering the variables of p and taking a suitable F-multiple of p, we may write

$$(1) \qquad p = x_1 x_2 x_3 + c_1 x_1 x_3 x_2 + c_2 x_2 x_1 x_3 + c_3 x_2 x_3 x_1 + c_4 x_3 x_1 x_2 + c_5 x_3 x_2 x_1.$$

Our first observation is that the derivations for p, which we call d, h, and k respectively, are all inner. The form of p and Lemma 7 show that d and k are inner derivations, with $r^d = ra - ar$ and $r^k = rt - tr$ for some $a, t \in R$. Using Litoff's Theorem, there are idempotents e and f so that $a, t \in eRe$ and $e, e^h \in fRf$. If $g \in \mathrm{Ann}(f)$ then $g^h \in \mathrm{Ann}(e) \subset \mathrm{Ann}(a) \cap \mathrm{Ann}(t)$. Evaluate p by replacing x_1 with x^d, x_2 with g^h, and x_3 with y^k, for any $x, y \in R$, and then multiply the result on both sides by e to obtain $-axg^hyt - c_5tyg^hxa = 0$. Replacing x with ex gives $aRg^hRt = 0$, and the simplicity of R forces $g^h = 0$. Hence, we may apply Theorem 1 to h and the idempotent f and write $r^h = rb - br$ for some $b \in R$. For the remainder of the proof the letters a, b, and t will represent the elements of R which induce the derivations d, h, and k, respectively.

The next series of computations will show that (2) and (3) of the theorem hold. After obtaining these we shall show that if p is given as in equation (1), then $p = S_3(x_1, x_2, x_3)$. Using Litoff's Theorem once again, there is an idempotent e so that $a, b, t \in eRe$. Choose any idempotent $f \in \mathrm{Ann}(e)$ and evaluate p using $x_1 = x \in eRf$, $x_2 = y \in fRe$, and $x_3 = z \in eRe$. From equation (1) and the observations $x^d = -ax$, $y^h = yb$, and $x^dz^k = z^ky^h = 0$, one obtains $-axybz^k - c_3ybz^kax - c_4z^kaxyb = 0$, so right multiplication by e gives

$$(2) \qquad axyb(zt - tz) + c_4(zt - tz)axyb = 0.$$

Since $z \in eRe$ is arbitrary, using equation (2) we have first that $axybtRe \subset Rt + Rb$, and then that $RaxybtRe \subset Rt + Rb$. If $axybt = 0$ for all choices of x and y, then the simplicity of R yields $aRbt = 0$, and then $bt = 0$. On the other hand if $bt \neq 0$ then $Re \subset Rt + Rb$ follows from the simplicity of R, for every idempotent e satisfying $a, b, t \in eRe$. But $\dim(Rt + Rb)$ is fixed, and by Litoff's Theorem one can find $a, b, t \in eRe$ with $\dim(Re)$ as large as desired. Therefore, we must have $bt = 0$.

Using $bt = 0$, reduces equation (2) to

$$(3) \qquad axybzt + c_4(zt - tz)axyb = 0.$$

Suppose that $c_4 = 0$. It would follow that $aRbRt = 0$, so one of a, b, or t would be zero. Since none of these is zero, $c_4 \neq 0$, and the free choice of $z \in eRe$ in equation (3) shows that $eRtaxyb \subset aR + tR$. As above, we must have $ta = 0$ and this reduces equation (3) to $axybzt = c_4tzaxyb$. It is now easy to see that $r(t) = r(b)$ and $l(a) = l(t)$. In particular, $t \in l(r(t)) = l(r(b)) = Rb$, and similarly, $b \in Rt$, resulting in $Rb = Rt$. Using $l(a) = l(t)$ gives $aR = tR$.

Now evaluate p, as in equation (1), with the new choices of $x_1 = x \in eRe$, $x_2 = y \in eRf$, and $x_3 = z \in fRe$, where e and f are as above. The result is

$$(4) \qquad -x^dbyzt - c_3byztx^d - c_4ztx^dby = 0$$

and right multiplication by f gives $c_4zt(xa - ax)by = 0$. But we have shown that $c_4 \neq 0$ and that $ta = 0$, so it follows that $ab = 0$, and equation (4) reduces to

$$(5) \qquad axbyzt = c_3byztxa.$$

Arguments like those in the last paragraph show that $c_3 \neq 0$, that $Rt = Ra$, and that $aR = bR$. Combining the information obtained so far shows that $c_3 \neq 0$, $c_4 \neq 0$, $aR = bR = tR$, $Ra = Rb = Rt$, and $bt = ta = ab = 0$. It follows that any product of two elements from $\{a, b, t\}$ is zero, so in particular, $a^2 = 0$.

Using equation (5), the choices represented by x, y, and z, and the simplicity of R, one has $axbyt = c_3bytxa$ for any $x, y \in R$. Since $c_3 \neq 0$, [3; Lemma 1.3.2, p. 22] can be applied to conclude that $aRb \subset Fb$. Consequently, aR is a minimal right ideal of R, so of course Ra is a minimal left ideal. Write $aR = vR$ and $Ra = Rg$, for v and g idempotents, so that $v = ar = vagrv$ for some $r \in R$. Now $b \in vR \cap Rg$, so $b = vbg = vagrvbg \in vagRg = ca$ for some $c \in F$, because $a = vag$ and $gRg = gF$. In the same way one shows that $t \in Fa$, which proves (2) and (3) of the theorem.

To complete the proof we need only show that $p = S_3(x_1, x_2, x_3)$. From what was proven above, there is no loss of generality in assuming $d = h = k$, since p is multilinear. Substituting in p, as given in equation (1), with $x_1 = x$, $x_2 = y$, and $x_3 = z$ for $x, y, z \in R$, we may write

$$(xa - ax)(y^dz^d + c_1z^dy^d) + y^dp_1 + z^dp_2 = 0.$$

Thus, for y and z fixed, $xa(y^dz^d + c_1z^dy^d) \in aR + y^dR + z^dR = T$, for every $x \in R$. Since $\dim(T)$ is finite and $Ra(y^dz^d + c_1z^dy^d)R \subset T$, we must conclude that $a(y^dz^d + c_1z^dy^d) = 0$ for each choice of $y, z \in R$. By choosing $z = y$ and using $a^2 = 0$, this expression reduces to $(1 + c_1)ayaya = 0$, which means that $(1 + c_1)aR$ is a nil right ideal of R. Therefore, we must conclude that $c_1 = -1$.

As in the second paragraph above, using equation (5), and the fact that $b, t \in Fa$, one can write $axaya = c_3ayaxa$ for any $x, y \in R$. Thus, $(1 - c_3)axaxa = 0$, which forces $c_3 = 1$. We have just shown that if $p = p(x_1, x_2, x_3)$ as in equation (1), then $c_1 = -1$. Since $c_3 = 1$, if we view $p = p(x_2, x_3, x_1)$, then by symmetry, $c_2 = -1$. The same observations, starting with equation (3), show first that $axbyt = c_4tyaxb$ for any $x, y \in R$, then that $c_4 = 1$, and finally, by considering $p = p(x_3, x_1, x_2)$, that $c_5 = -1$. Consequently, $p = S_3(x_1, x_2, x_3)$, completing the proof of the theorem.

Theorem 2 shows that, essentially, the only PDI of minimal type for a special ring is $S_3(x, y, z)$. We want to argue that this same basic result holds for a PDI of minimal type satisfied by any prime non-PI ring. Of course one cannot expect the derivations will be inner, as is the case for special rings, as our next example shows.

EXAMPLE 3. Let J be the ring of integers, so that $Q(J)$ is the field of rational numbers. In the ring of row finite countable by countable matrices over $Q(J)$, let T be the subring of matrices having only finitely many nonzero entries. Clearly, T is a special ring, and so by Theorem 2, T satisfies $S_3(x^d, y^d, z^d)$ where $d = \mathrm{ad}(e_{12})$, the inner derivation induced by the matrix unit e_{12}. If

$$R = \{(a_{ij}) \in T \mid a_{ij} \in 2J \text{ for all } i \text{ and } j\},$$

then R is a prime ring with extended centroid $C = Q(J)$, and $RC = T$. Now $d \in \mathrm{Der}(R)$ and $R \subset T$, so $S_3(x^d, y^d, z^d)$ is a PDI for R. Although d is not an inner derivation of R, $2d$ is an inner derivation of R.

Let R be a non-PI prime ring satisfying a PDI p of minimal type. As in the beginning of the proof of Theorem 2, by substituting zero for sets of variables in p one can write $p = p_1 + \cdots + p_t$, where each p_i is a PDI for R of the same type as p and each variable in p_i appears in each monomial of p_i. Assume for now that $p = p_1$. Let q be the linearization of p. The comments after Lemma 1 and Theorem 2 show that $q = S_3(x, y, z)$, regarded as a PDI for $H \otimes_C F$, where C is the extended centroid of R, $H = \mathrm{Soc}(RC)$, and F is an algebraic closure of C. The argument in the beginning of Theorem 2 shows that $p = q = S_3(x, y, z)$. Therefore $(3, 3)$ is the minimal type for PDIs and any such for any non-PI prime ring R is a C-linear combination of PDIs $S_3(x, y, z)$. What remains is to describe the derivations which can occur in this general case.

Assume $p = S_3(x, y, z)$ is a PDI for R. Then as a PDI for $H \otimes F$, as above, its derivations as elements in $\mathrm{Der}(H \otimes F)$ are described in Theorem 2. Let d be the derivation acting on x, so that d is commutation with $a = \Sigma\, h_i \otimes f_i \in H \otimes F$, where we may assume that $\{f_i\}$ is C-independent. Identify R as $R \otimes 1_F$ and use the definition of RC to find an ideal I of R so that $Ih_1 \subset R$ and is nonzero. It follows that $R \cap (H \otimes F) \neq 0$ so for any $t \in R \cap (H \otimes F)$

$$t^d = ta - at = \sum\, [t, h_i] \otimes f_i \in R \cap (H \otimes F),$$

since $d \in \mathrm{Der}(R)$. Using the C-independence of $\{f_i\}$, we may conclude that either $a \in H$, or that $[t, h_i] = 0$ for $i > 1$, and $f_1 \in C$. The second possibility would mean that h_i centralizes the nonzero ideal $R \cap (H \otimes F)$ of R, and so would force $h_i \in H \cap C = 0$ for $i > 1$. Therefore, we must have $a \in H$. Now the derivations acting on y and on z must each have the form $\mathrm{ad}(a \otimes f)$ for some $f \in F$ by Theorem 2, so by the argument just given $f \in C$, and these derivations are in Cd.

Of course we know that $a^2=0$ and that aRC is a minimal right ideal from Theorem 2. As in Example 3, the best one can do, in describing d in terms of R, is to show that $Ca \cap R \neq 0$. This would show that a suitable C-multiple of p, and so of d, is a PDI for which the derivations are inner. Our final description of PDIs of minimal type will be an immediate consequence of our discussion above and our next theorem which shows that H is, essentially, a central localization. This result can be found in [10; Proposition 7.6, p. 50, and Theorem 7.7, p. 51], but we give a fairly easy argument for the sake of completeness. Recall that if $Z(A)=Z$ is the center of a prime ring A, then AZ^{-1} is the localization of A at the nonzero elements of Z.

THEOREM 3. *Let R be a prime ring which satisfies a generalized polynomial identity, set $RC = W$, and let $e \in \mathrm{Soc}(W)$ be a nonzero idempotent. The following hold:*

(1) *$D = R \cap eWe$ is a prime ring;*
(2) *$eWe = DZ(D)^{-1}$;*
(3) *$eC = QF(Z(D))$;*
(4) *$\mathrm{Soc}(W) = (R \cap \mathrm{Soc}(W))C$; and*
(5) *for any $h_1, \dots, h_n \in \mathrm{Soc}(W)$, there is $c \in C$ so that $h_ic \in R - \{0\}$ for each i.*

PROOF. Since R satisfies a generalized polynomial idenitty, $\mathrm{Soc}(W) \neq 0$ and eWe is a simple algebra finite dimensional over its center, eC [8]. For any nonzero ideal I of R, $eICe$ is a nonzero ideal of eWe, so $eICe = eWe$ must hold. Let V be a nonzero ideal of R satisfying $0 \neq Ve + eV \subset R$, and set $I = V^2$. Now $eIe \subset eWe \cap R = D$, so it follows that $eWe = eICe \subset DC \subset eWe$. Therefore, $eWe = DC$, proving (1), and also that $Z(D) \subset eC$. The finite dimensionality of eWe over its center shows that D satisfies a polynomial identity, from which we may conclude that $Z(D) \neq 0$ and that $DZ(D)^{-1}$ is a simple algebra, finite dimensional over its center, $QF(Z(D))$ [9]. But $Z(D) \subset eC$, so $eWe = DC = (DZ(D)^{-1})C$ and the identity element of $DZ(D)^{-1}$ is a central idempotent in eWe, forcing it to be e. To prove (3), let $c \in C - \{0\}$, T an ideal of R with $0 \neq Tc \subset R$, and set $B = (V \cap T)^3$. As above, eBe is an ideal of D, so $DZ(D)^{-1} = eBeZ(D)^{-1}$ so we may write $e = ebez^{-1}$ for some $b \in B$ and $z \in Z(D)$. Consequently, $ce = ecbez^{-1} \in DZ(D)^{-1} \cap Z(eWe)$, and it follows that $eC \subset QF(Z(D))$. This and the opposite conclusion obtained above gives (3), and also (2) since $eWe = DC$.

Finally, given $h_1, \dots, h_n \in \mathrm{Soc}(W)$, use Litoff's Theorem to find $e \in \mathrm{Soc}(W)$ satisfying $\{h_i\} \subset eWe$. Using (2), write $h_i = r_iz^{-1}$ for $r_i \in R \cap eWe$ and $z \in Z(R \cap eWe)$, with $z = ec$ for some $c \in C$ by (3). Clearly, for each i,

$h_i c = h_i ec = h_i z \in R$ proving (5). Since (4) is a special case of (5), the theorem is proved.

Theorem 3 and our earlier comments enable us to state our main result for prime rings.

THEOREM 4. *Among all PDIs satisfied by some non-PI prime ring, let p be of minimal type (k, n) and be satisfied by R. Then $k = n = 3$, p is linear in each of its variables, and $p = p_1 + \cdots + p_m$ where each p_i is a multilinear PDI for R of type $(3,3)$ and satisfies*:

(1) $p_i = c_i S_3(x_i, y_i, z_i)$ *for* $c_i \in C$; *and*

(2) *the derivations associated with p_i are all equal to* $\mathrm{ad}(a)$ *for $a \in R$ so that $a^2 = 0$ and aRC is a minimal right ideal in RC.*

PROOF. The discussion preceding Theorem 3 shows that $p = c_1 p_1 + \cdots + c_m p_m$ where each $p_i = S_3(x_i, y_i, z_i)$ is a PDI for R with corresponding derivations d_i, h_i and k_i C-dependent and inner on $\mathrm{Soc}(RC)$. Also, if these derivations are, respectively, $\mathrm{ad}(w)$, $\mathrm{ad}(wf_1)$, and $\mathrm{ad}(wf_2)$ for $f_i \in C$, then $w^2 = 0$ and wRC is a minimal right ideal in RC. By Theorem 3, there is some $t \in C - \{0\}$, so that $tw \in R$. The multilinearity of $S_3(x_i, y_i, z_i)$ yields $S_3(x_i^{d_i}, y_i^{h_i}, z_i^{k_i}) = f_1 f_2 t^{-3} S_3(x_i^d, y_i^d, z_i^d)$ for $d = \mathrm{ad}(tw)$, proving the theorem.

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